Lecture 20:

More Complex Models

Background:

In the previous arrival model, we assume the arrival rate is a constant λ. In a more complicated case, the arrival rate may vary on time, i.e., $\lambda = \lambda(t)$. Moreover, different students might have various arrival probabilities. In this lecture, assume the total number of students is n and for each $1 \leq m \leq n$, the arrival probability is $p_{n,m}$. Thus, one can use $X_{n,m} \sim \text{Bernoulli}(p_{n,m}) = \text{binomial}(1, p_{n,m})$ to represent the arrival event for m-th student of the total n-student case.

Definition 20.1. (Total Variation norm and Total Variation distance)

Suppose that μ is a signed measure on (Ω, Σ).

$$\|\mu\|_{TV} := \sup\{\mu(A) \mid A \in \Sigma\} - \inf\{\mu(A) \mid A \in \Sigma\}.$$

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Suppose that X & Y are integer-valued random variables,

$$d_{TV}(X,Y) = \max_{A \subseteq \mathbb{Z}} |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)|.$$

Remark 20.1. $d_{TV}(X,Y) = \|X-Y\|_{TV} = \frac{1}{2} \underbrace{\sum_k |\mathbb{P}(X=k) - \mathbb{P}(Y=k)|}_{\|X-Y\|_1 :\ L_1 \text{ distance between } X \text{ and } Y.}$

Remark 20.2. One can check that d_{TV} is a metric

① $d_{TV}(X,Y) \geq 0$; and "=" holds iff $X=Y$;

② $d_{TV}(X,Y) = d_{TV}(Y,X)$;

③ $d_{TV}(X,Y) \leq d_{TV}(X,Z) + d_{TV}(Z,Y)$.

Definition 20.2. (Kullback-Leibler divergence)

Suppose μ and ν are two probability distributions defined on the space $\mathcal{X}$, then

$$D_{KL}(\mu \| \nu) = \sum_{x \in \mathcal{X}} \mu(x) \cdot \log \frac{\mu(x)}{\nu(x)}.$$

Remark 20.3. (Pinsker's Inequality). $\|\mu - \nu\|_{TV} \leq \sqrt{\frac{1}{2} D_{KL}(\mu \| \nu)}$.

Definition 20.3. (Convergence in TV norm). We say random variables $X_n \longrightarrow X$ in TV if

$$\lim_{n\to\infty} \| X_n - X \|_{TV} = 0.$$

Theorem 20.1. Suppose $X_{n,m} \sim \text{Bernoulli}(p_{n,m})$ are independent. Let $S_n = \sum_{m=1}^{n} X_{n,m}$, $\lambda_n = \mathbb{E} S_n = \sum_{m=1}^{n} p_{n,m}$, and $Y_n = \text{Poisson}(\lambda_n)$. Then

$$\|S_n - Y_n\|_{TV} \leq \sum_{m=1}^{n} p_{n,m}^2.$$

If, moreover, $\sup_m p_{n,m} \xrightarrow{n\to\infty} 0$, $\lambda_n \xrightarrow{n\to\infty} \lambda$, and $Y = \text{Poisson}(\lambda)$, then

$$\lim_{n\to\infty} \|S_n - Y\|_{TV} = 0,$$

i.e., $S_n \longrightarrow \text{Poisson}(\lambda)$ in TV norm.

Proof.
$$\|S_n - Y_n\|_{TV}$$
$$= \Big\| \sum_{m=1}^{n} X_{n,m} - \sum_{m=1}^{n} \text{Poisson}(p_{n,m}) \Big\|_{TV}$$
$$= \Big\| \sum_{m=1}^{n} \big(X_{n,m} - \text{Poisson}(p_{n,m}) \big) \Big\|_{TV}$$

$$\leq \sum_{m=1}^{n} \| X_{n,m} - \mathrm{Poisson}(p_{n,m}) \|_{TV}$$

$$= \sum_{m=1}^{n} \frac{1}{2} \sum_{k=0}^{\infty} \left| \mathbb{P}(X_{n,m} = k) - \mathbb{P}(\mathrm{Poisson}(p_{n,m}) = k) \right|$$

$$= \frac{1}{2} \sum_{m=1}^{n} \Big\{ \left| (1-p_{n,m}) - e^{-p_{n,m}} \right| + \left| p_{n,m} - p_{n,m} e^{-p_{n,m}} \right| + \sum_{k=2}^{\infty} \mathbb{P}(\mathrm{Poisson}(p_{n,m}) = k) \Big\}$$

$\boxed{1 \geq e^{-p_{n,m}} \geq 1 - p_{n,m}}$

$$= \frac{1}{2} \sum_{m=1}^{n} \Big\{ e^{-p_{n,m}} - 1 + p_{n,m} + p_{n,m} - p_{n,m} e^{-p_{n,m}} + 1 - e^{-p_{n,m}} - p_{n,m} e^{-p_{n,m}} \Big\}$$

$$= \sum_{m=1}^{n} p_{n,m} \left(1 - e^{-p_{n,m}} \right)$$

$$\leq \sum_{m=1}^{n} p_{n,m}^2$$

Notice that $\| Y_n - Y \|_{TV}$

Pinsker's inequality

$$\leq \sqrt{\frac{1}{2} D_{KL}(Y_n \| Y)}$$

why?

$$= \sqrt{\frac{1}{2} \left(\lambda - \lambda_n + \lambda_n \cdot \log \frac{\lambda_n}{\lambda} \right)}$$

$\boxed{\log x \leq x - 1}$

$$\leq \sqrt{\frac{1}{2} \left(\lambda - \lambda_n + \lambda_n \cdot \frac{\lambda_n - \lambda}{\lambda} \right)}$$

$$= \frac{|\lambda_n - \lambda|}{\sqrt{2\lambda}} .$$

Thus, by the triangular inequality,

$$
\begin{aligned}
0 &\le \|S_n - Y\|_{TV} \\
&\le \|S_n - Y_n\|_{TV} + \|Y_n - Y\|_{TV} \\
&\le \sum_{m=1}^{n} p_{n,m}^2 + \frac{|\lambda_n - \lambda|}{\sqrt{2\lambda}} \\
&\le \left(\max_{1\le m\le n} p_{n,m}\right)\cdot\left(\sum_{m=1}^{n} p_{n,m}\right) + \frac{|\lambda_n - \lambda|}{\sqrt{2\lambda}} \\
&= \left(\max_{1\le m\le n} p_{n,m}\right)\cdot \lambda_n + \frac{|\lambda_n - \lambda|}{\sqrt{2\lambda}} .
\end{aligned}
$$

Since $\lim_{n\to\infty}\left(\max_{1\le m\le n} p_{n,m}\right) = 0$ and $\lim_{n\to\infty} \lambda_n = \lambda$, one has $\lim_{n\to\infty}\left(\max_{1\le m\le n} p_{n,m}\right)\cdot\lambda_n + \frac{|\lambda_n - \lambda|}{\sqrt{2\lambda}} = 0$.

By the squeeze theorem, we know

$$\lim_{n\to\infty} \|S_n - Y\|_{TV} = 0. \quad \square$$

Example 20.1. Let $p_{n,m} = \frac{\lambda}{n}$, $\lambda_n = \lambda$, thus

$$\max_m p_{n,m} = \frac{\lambda}{n} \xrightarrow{n\to\infty} 0 \quad \text{and} \quad \lambda_n \xrightarrow{n\to\infty} \lambda .$$

By Theorem 20.1,

$$S_n = X_{n,1} + X_{n,2} + \cdots + X_{n,n} \xrightarrow{TV} \text{Poisson}(\lambda).$$

This implies Theorem 18.1. Thus, one can view Theorem 18.1 as a special case of Theorem 20.1.

Example 20.2. Suppose $p_{n,m}$ is not a constant but $\max_{1\le m\le n} p_{n,m} \xrightarrow{n\to\infty} 0$ and $\lambda_n \xrightarrow{n\to\infty} \lambda$. Then

$$S_n \xrightarrow{TV} \text{Poisson}(\lambda).$$

Example 20.3. Let $\lambda : [0,1] \longrightarrow \mathbb{R}_+$ be continuous and $p_{n,m} = \frac{1}{n}\lambda(\frac{m}{n})$, $\forall n \in \mathbb{N}$, $\forall 1 \le m \le n$. Then

$$0 \le \max_{1\le m\le n} p_{n,m} \le \frac{1}{n} \max_{t\in[0,1]} \lambda(t) \xrightarrow{n\to\infty} 0,$$

and $\lambda_n = \sum_{m=1}^{n} \frac{1}{n}\lambda(\frac{m}{n}) \xrightarrow{n\to\infty} \int_0^1 \lambda(t)\,dt$. Thus,

$$S_n \longrightarrow \text{Poisson}\left(\int_0^1 \lambda(t)\,dt\right).$$

Definition 20.4. (Non-homogeneous Poisson Process)

Let $N(t)$ represents the total number of occurance or events that have happened up to and including time t. We say that

$\{N(s),\ s\geq 0\}$ is a non-homogeneous Poisson Process with rate $\lambda(r)$ if

(i). $N(0)=0$,

(ii). $N(t+s)-N(s) = \text{Poisson}\left(\int_s^{t+s}\lambda(r)\,dr\right)$, and

(iii). $N(t)$ has independent increments.

Remark 20.4. Interarrivals are in general neither exponential distributions nor independent. For instance,

$$\mathbb{P}(\tau_1>t) = \mathbb{P}(N(t)=0) = \mathbb{P}\left(\text{Poisson}\left(\int_0^t\lambda(r)\,dr\right)=0\right)$$

$$= e^{-\int_0^t \lambda(r)\,dr}$$

is not exponential unless λ is a constant.

This is the end of this lecture!